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Effect of a General Body Force on a 2D Thermoelastic Long Cylinder under Green-Lindsay Theory

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Abstract: **We consider a 2D infinite cylindrical thermoelastic body. The theory used is that of thermoelasticity due to Green – Lindsay. The surface is assumed to be traction free and subjected to a known asymmetric temperature distribution. The body is under the action of a general body force. Using Fourier series, we develop a general solution for any type of body forces. We next apply our method to a specific problem. Laplace transform is used. The inversion process is carried out numerically. Numerical results are computed for the temperature, displacement and stress distributions and shown graphically.**

Keywords: **Body forces; Laplace transform; Long cylinder; Non-Solenoidal; Thermoelasticity due to Green – Lindsay; Two relaxation time.**

1. INTRODUCTION

Biot [1] developed the theory of coupled thermoelasticity this theory was found to deviate from physical realities in that it predicts infinite speed of propagation for thermal waves. Lord and Shulman [2] were the first to develop a theory of thermoelasticity that ensures finite wave speeds. Their theory is called the theory of thermoelasticity with one relaxation time. They have obtained their theory by modifying the Fourier's law of heat conduction. Some contributions to the subject can be found in [3-8]. Green and Lindsay [9-10] derived the governing equations of the theory of thermoelasticity with two relaxation times. They have used a generalization of a known thermodynamic inequality. Their theory does not violate Fourier's law when the body has a centre of symmetry. Some contributions to this theory are [11-16].

In industry, the effect of body forces is very important, to the authors' knowledge, all the papers dealing with body forces in the generalized theory of thermoelasticity choose to deal with solenoidal forces only to simplify the governing equations [17-18].

In this manuscript, we show how to deal with non solenoidal body forces. Our treatment can also be used for solenoidal body forces.

2. FORMULATION OF THE PROBLEM

In this work we consider a two-dimensional problem for an infinite long cylinder with radius "a" within the context of the theory of thermoelasticity with two relaxation times**.** We consider a homogeneous isotropic thermoelastic solid occupying the infinite circular cylinder region $0 \le r \le a$, $0 \le \varphi \le 2\pi$, $-\infty \le z \le \infty$, where (r, φ, z) are cylindrical polar coordinates.

We shall also assume that the initial state of the medium is quiescent. The surface of the cylinder is traction-free and enclosed in a surrounding medium with temperature distribution varying circumferentially and under the action of body forces.

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From the physics of the problem, it is clear that all the functions considered will depend on r, φ and t only.

The displacement vector **u**, thus, has the components

$$u_r = u(r,\varphi,t)\ ,\ u_\varphi = v(r,\varphi,t) \text{ and } u_z(r,\varphi,t) = 0 \tag{1}$$

The components of the strain tensor are thus given by

$$e_{rr} = \frac{\partial u}{\partial r} \tag{2a}$$

$$e_{\varphi\varphi} = \frac{1}{r}\frac{\partial v}{\partial \varphi} + \frac{u}{r} \tag{2b}$$

$$e_{r\varphi} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u}{\partial \varphi} + \frac{\partial v}{\partial r} - \frac{v}{r}\right) \tag{2c}$$

$$e_{zz} = e_{rz} = e_{\phi z} = 0 \tag{2d}$$

The cubical dilatation e is thus given by

$$\bar{\theta}(a,\varphi,t) = \bar{f}(\varphi,s) \tag{3}$$

The equations of motion can be written as

$$(\lambda+\mu)\ \text{grad}\ e + \mu\nabla^2\mathbf{u} - \gamma(\ \text{grad}\, T + \upsilon\frac{\partial}{\partial t} grad\ T) + \mathbf{F} = \rho\,\frac{\partial^2\mathbf{u}}{\partial t^2} \tag{4}$$

The generalized equation of heat conduction has the form

$$k\nabla^2 T = \frac{\partial}{\partial t}\left(1+\tau\frac{\partial}{\partial t}\right)\left[\rho\, C_E T + \gamma\, T_0\,\frac{\partial e}{\partial t}\right] \tag{5}$$

In the above equations, T is the absolute temperature and e is the cubical dilatation given by $e = \frac{1}{r}\left(\frac{\partial(r\,u)}{\partial r} + \frac{\partial v}{\partial \varphi}\right)$.

In the preceding equations ρ is the density, the constants λ and μ are Lamé's constants and $\gamma = \alpha_t(3\lambda + 2\mu)$ where α_t is the coefficient of linear thermal expansion. C_E is the specific heat at constant strain, k is the thermal conductivity, τ and ν are the relaxation times and T_0 is a reference temperature assumed to be such that $|(T-T_0)/T_0| << 1$. $\mathbf{F} = (F_r, F_\varphi, 0)$ is the body forces vector per unit volume and ∇^2 is Laplace's operator given in our case by :

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$$

The components of the stress tensor are given by

$$\sigma_{ij} = 2\mu e_{ij} + [\lambda e_{kk} - \gamma(T - T_0 + \nu\frac{\partial T}{\partial t})]\delta_{ij} \tag{6}$$

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For convenience, we shall use the following non-dimensional variables:

$$r^* = c_1\eta r, \quad \mathbf{u}^* = c_1\eta \mathbf{u}, \quad v^* = c_1\eta v, \quad t^* = c_1^2\eta t, \quad \tau^* = c_1^2\eta\tau, \quad \upsilon^* = c_1^2\eta\upsilon$$

$$F_r^* = \frac{F_r}{(\lambda+2\mu)c_1\eta}, \quad F_\phi^* = \frac{F_\phi}{(\lambda+2\mu)c_1\eta}, \quad \sigma_{ij}^* = \frac{\sigma_{ij}}{\mu}, \quad \theta = \frac{\gamma(T-T_0)}{(\lambda+2\mu)},$$

where $\eta = \dfrac{\rho C_E}{k}$, $c_1 = \sqrt{\dfrac{\lambda+2\mu}{\rho}}$. c_1 is the speed of propagation of isothermal longitudinal elastic waves.

Using the above non-dimensional variables, the governing equations take the form

$$\left(\beta^2-1\right)\ \text{grad e} + \nabla^2\mathbf{u} - \beta^2\ \text{grad}\,\theta - \beta^2\upsilon\frac{\partial}{\partial t}(\text{grad}\,\theta) + \beta^2\mathbf{F} = \beta^2\ \frac{\partial^2\mathbf{u}}{\partial t^2} \tag{7}$$

$$\nabla^2\theta = \left(\frac{\partial}{\partial t} + \tau\frac{\partial^2}{\partial t^2}\right)\theta + \varepsilon\frac{\partial e}{\partial t} \tag{8}$$

while the constitutive relation (6) becomes

$$\sigma_{ij} = [2e_{ij} + (\beta^2-2)e_{kk}\delta_{ij} - \beta^2(1+v\frac{\partial}{\partial t})\theta\delta_{ij}] \tag{9}$$

where $\beta^2 = \dfrac{\lambda+2\mu}{\mu}$, $\varepsilon = \dfrac{T_0\gamma^2}{\rho C_E(\lambda+2\mu)}$.

Using the vector identity

$$\nabla^2\mathbf{u} = grad\ div\,\mathbf{u} - curl\,curl\,\mathbf{u} = grad\,e - curl\,curl\,\mathbf{u} \tag{10}$$

Then equation (7) takes the form

$$\beta^2\ \text{grad e} - \text{curl curl}\mathbf{u} - \beta^2(1+\upsilon\frac{\partial}{\partial t})\text{grad}\,\theta + \beta^2\mathbf{F} = \beta^2\ \frac{\partial^2\mathbf{u}}{\partial t^2} \tag{11}$$

Applying the div operator to both sides of the above equation, we get

$$\left(\nabla^2 - \frac{\partial^2}{\partial t^2}\right)e + \text{div}\mathbf{F} = \left(1+\upsilon\frac{\partial}{\partial t}\right)\nabla^2\theta \tag{12}$$

The boundary conditions are taken as

$$\theta\left(a,\varphi,t\right) = f\left(\varphi,t\right) \tag{13a}$$

$$\sigma_{rr}\left(a,\varphi,t\right) = 0 \tag{13b}$$

$$\sigma_{r\varphi}\left(a,\varphi,t\right) = 0 \tag{13c}$$

where $f(\varphi,t)$ is a known function.

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3. SOLUTION IN THE LAPLACE TRANSFORM DOMAIN

Applying the Laplace transform defined by the relation [19]

$$\varphi\ \bar{f}(r,\varphi,s)=\mathrm{L}[f(r,\phi,t)]=\int_0^\infty f(r,\varphi,t)\,e^{-st}\,dt,$$

to both sides of Eqs. (3), (8), (9), (11) and (12) we obtain

$$\bar{e}=\frac{1}{r}\left(\frac{\partial(r\bar{u})}{\partial r}+\frac{\partial\bar{v}}{\partial\varphi}\right) \tag{14}$$

$$(\nabla^2-s-\tau s^2)\bar{\theta}=\varepsilon s\bar{e} \tag{15}$$

$$\bar{\sigma}_{ij}=[2\bar{e}_{ij}+(\beta^2-2)\bar{e}\,\delta_{ij}-\beta^2(1+\nu s)\bar{\theta}\,\delta_{ij}] \tag{16}$$

$$\beta^2\,\mathrm{grad}\ \bar{e}-curl\,curl\,\bar{\mathbf{u}}-\beta^2(1+\upsilon s)grad\ \bar{\theta}+\beta^2\bar{\mathbf{F}}=\beta^2 s^2\bar{\mathbf{u}} \tag{17}$$

$$\left(\nabla^2-s^2\right)\bar{e}+div\bar{\mathbf{F}}=\left(1+\upsilon s\right)\nabla^2\bar{\theta} \tag{18}$$

The boundary conditions (13), in the transformed domain, take the form

$$\bar{\theta}\left(a,\varphi,t\right)=\bar{f}\left(\varphi,s\right) \tag{19a}$$

$$\bar{\sigma}_{rr}\left(a,\varphi,s\right)=0 \tag{19b}$$

$$\bar{\sigma}_{r\varphi}\left(a,\varphi,s\right)=0 \tag{19c}$$

Eliminating $\bar{e}$ between Eqs.(15) and (18), we get

$$[\nabla^4-\left(s^2(1+\tau)+\varepsilon s(1+\upsilon s)+s\right)\nabla^2+s^3(1+\tau s)]\bar{\theta}=-\varepsilon s\,div\,\bar{\mathbf{F}} \tag{20}$$

The above equation can be factorized as

$$(\nabla^2-k_1^2)(\nabla^2-k_2^2)\bar{\theta}=-\varepsilon s\,div\,\bar{\mathbf{F}} \tag{21}$$

where k_1^2 and k_2^2 are the roots with positive real parts of the characteristic equation

$$k^4-\left(s^2(1+\tau)+\varepsilon s(1+\upsilon s)+s\right)k^2+s^3(1+\tau s)=0 \tag{22}$$

The complementary solution of Eq.(21) can be written in the form

$$\bar{\theta}=\bar{\theta}_c+\bar{\theta}_p$$

$\bar{\theta}_c=\bar{\theta}_{1c}+\bar{\theta}_{2c}$ where $\bar{\theta}_{ic}$ is the solution of the homogeneous equation

$$(\nabla^2-k_i^2)\bar{\theta}_{ic}=0,\quad i=1,2 \tag{23}$$

We shall solve Eq. (23) by the method of Eigen function expansion [20-21]. We write the function $\bar{\theta}_{\mathrm{C}}(\mathrm{r},\varphi,\mathrm{s})$ in the form

$$\bar{\theta}_{\mathrm{C}}(\mathrm{r},\varphi,\mathrm{s})=\mathrm{R}(\mathrm{r},\mathrm{s})\ \Phi(\varphi) \tag{24}$$

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Substituting from Eq.(24) into Eq.(23) then we get

$$\frac{1}{R}\left(r^2\frac{\partial^2 R}{\partial r^2}+r\frac{\partial R}{\partial r}\right)-k_i^2r^2+\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\varphi^2}=0 \tag{25}$$

Since the variables r, s and φ are independent, we obtain the two equations

$$\frac{1}{R}\left(r^2\frac{\partial^2 R}{\partial r^2}+r\frac{\partial R}{\partial r}\right)-k_i^2r^2=C \tag{26a}$$

$$\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\varphi^2}=-C. \tag{26b}$$

where C is a constant.

Since $\bar{\theta}_c$ and hence Φ are periodic functions in φ of period 2π, we must have

$$C=n^2,\quad n=0,1,2,..... \tag{27}$$

Substituting from Eq. (27) into Eq. (26), we obtain

$$r^2\frac{\partial^2 R}{\partial r^2}+r\frac{\partial R}{\partial r}-(k_i^2r^2+n^2)R=0, \tag{28a}$$

$$\frac{\partial^2\Phi}{\partial\varphi^2}=-n^2\Phi \tag{28b}$$

The solutions of Eqs.(28) have the form

$$R=R_{ni}(r,s)=I_n(k_ir),K_n(k_ir),\quad n=0,1,2,..... \tag{29a}$$

$$\Phi=\Phi_n(\varphi)=\cos(n\varphi),\sin(n\varphi),\quad n=0,1,2,..... \tag{29b}$$

where $I_n(k_ir)$ and $K_n(k_ir)$ are the modified Bessel functions of the first and second kinds of order n, respectively.

Since we are seeking solutions that are bounded at the origin, then $K_n(k_ir)$ are not appropriate Eigen functions for our problem, we thus obtain

$$\mathrm{R}=\mathrm{R}_{ni}(\mathrm{r,s})=\mathrm{I}_n(\mathrm{k_ir}),\quad i=1,2,\quad n=0,1,2,.... \tag{30a}$$

We shall also take the function $\Phi_n(\varphi)$ to be an even function of φ, thus

$$\Phi=\Phi_n(\varphi)=\cos(n\varphi),\quad n=0,1,2,..... \tag{30b}$$

The general solution of Eq. (23) is given by

$$\bar{\theta}_c=\sum_{n=0}^{\infty}\sum_{i=1}^{2}A_{ni}(k_i^2-s^2)I_n(k_ir)\cos(n\varphi) \tag{31}$$

where $A_{ni}, i=1,2,\quad n=0,1,2,....$ are some parameters depending on s only.

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The particular solution depends on the form of $\overline{\mathbf{F}}$.

The general solution of equation (21) become

$$\bar{\theta} = \sum_{n=0}^{\infty}\sum_{i=1}^{2} A_{ni}(k_i^2 - s^2)\mathrm{I}_n(k_i r)\cos(n\varphi) + \bar{\theta}_p \tag{32}$$

Similarly, it can be shown that

$$\bar{e} = (1+\upsilon s)\sum_{n=0}^{\infty}\sum_{i=1}^{2} A_{ni}k_i^2 I_n(k_i r)\cos(n\varphi) + \bar{e}_p \tag{33}$$

Now, we have

$$curl\,\bar{\mathbf{u}} = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\varphi & \hat{e}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \varphi} & \dfrac{\partial}{\partial z} \\ u & rv & 0 \end{vmatrix} = \frac{1}{r}\left[\frac{\partial(rv)}{\partial r} - \frac{\partial u}{\partial \varphi}\right]\hat{e}_z \tag{34}$$

and

$$curl\ curl\,\bar{\mathbf{u}} = \left[\frac{1}{r^2}\frac{\partial}{\partial\varphi}\left(\frac{\partial(rv)}{\partial r} - \frac{\partial u}{\partial\varphi}\right)\right]\hat{e}_r - \left[\frac{\partial}{\partial r}\left(\frac{1}{r}\left(\frac{\partial(rv)}{\partial r} - \frac{\partial u}{\partial\varphi}\right)\right)\right]\hat{e}_\varphi \tag{35}$$

Substituting from Eq's (34) and (35) into Eq. (17) and equating the coefficient of $\hat{e}_r$, we get

$$\left(\nabla^2 - \beta^2 s^2\right)(r\bar{u}) = 2\bar{e} - \beta^2 r\bar{F}_r - r\frac{\partial}{\partial r}[(\beta^2 - 1)\bar{e} - \beta^2(1+\upsilon s)\bar{\theta}] \tag{36}$$

Substituting from Eqs. (32) and (33), we obtain the following equation satisfied by $\bar{u}$

$$\begin{aligned}\left(\nabla^2 - \beta^2 s^2\right)(r\bar{u}) &= 2\bar{e}_p - \beta^2 r\bar{F}_r - r\frac{\partial}{\partial r}\left[\left(\beta^2 - 1\right)\bar{e}_p - \beta^2(1+\upsilon s)\bar{\theta}_p\right] \\ &+ (1+\upsilon s)\sum_{n=0}^{\infty}\sum_{i=1}^{2} A_{ni}\{[2k_i^2 + n(k_i^2 - \beta^2 s^2)]I_n(k_i r) - [k_i^2 - \beta^2 s^2]r\,k_i\,I_{n+1}(k_i r)\}\cos(n\varphi)\end{aligned} \tag{37}$$

In obtaining the above equation, we have used the following relations of the modified Bessel functions of the second kind [22, 23]

$$\begin{aligned}\frac{dI_n(k_i r)}{dr} &= \frac{n}{r}I_n(k_i r) + k_i I_{n+1}(k_i r) \\ \frac{dI_{n+1}(k_i r)}{dr} &= k_i I_n(k_i r) - \frac{n+1}{r}I_{n+1}(k_i r)\end{aligned} \tag{38}$$

The solution of Eq. (37) is the sum

$$\bar{u} = \bar{u}_c + \bar{u}_{p0} + \bar{u}_{pF} \tag{39}$$

The complementary function $\bar{u}_c$ is the solution of the homogeneous equation

$$\left(\nabla^2 - \beta^2 s^2\right)(r\bar{u}_c) = 0 \tag{40}$$

Following the same steps done in solving Eq. (23), we arrive at

$$\bar{u}_c = \frac{(1+\upsilon s)}{r} \sum_{n=0}^{\infty} B_n(s) I_n(\beta sr) \cos n\varphi \tag{41}$$

$\bar{u}_{p0}$ is the particular solution corresponding to the summation in the right hand side of equation (30) which can be determined without knowing the form of the body force as:

$$\bar{u}_{p0} = \frac{(1+\upsilon s)}{r} \sum_{n=0}^{\infty} \sum_{i=1}^{2} A_{ni} \{nK_n(k_i r) - r k_i K_{n+1}(k_i r)\} \cos(n\phi)$$

$\bar{u}_{\boldsymbol{p}F}$ is the particular solution corresponding to the body force which will be determined later on. Collecting the above results, we obtain

$$\bar{u} = \frac{(1+\nu s)}{r} \left[\sum_{n=0}^{\infty} B_n(s) I_n(\beta s r) \cos n\phi + \sum_{n=0}^{\infty} \sum_{i=1}^{2} A_{ni} \{nI_n(k_i r) + r k_i I_{n+1}(k_i r)\} \cos(n\phi) \right] + \bar{u}_{\boldsymbol{p}F} \tag{42}$$

Since we have:

$$\frac{\partial \bar{v}}{\partial \varphi} = r\,\bar{e} - \frac{\partial}{\partial r}(r\bar{u}). \tag{43}$$

Substituting from Eqs. (33) and (42) into Eq.(43) and integrating with respect to φ, we obtain

$$\bar{v} = -\frac{(1+\upsilon s)}{r} \left(\sum_{n=0}^{\infty} B_n [I_n(\beta s r) + \frac{\beta s r}{n} I_{n+1}(\beta s r)] \sin n\phi + \sum_{n=0}^{\infty} \sum_{i=1}^{2} A_{ni}\, n I_n(k_i r) \sin n\phi \right) + \bar{v}_{pF} \tag{44}$$

where $\bar{v}_{pF}$ is the particular solution given by

$$\bar{v}_{pF} = \int \left[r\bar{e}_{pF} - \frac{\partial}{\partial r}\left[r\bar{u}_{pF} \right] \right] d\phi \tag{45}$$

The stress components, the components in cylindrical polar coordinates have the form

$$\bar{\sigma}_{rr} = 2\frac{\partial \bar{u}}{\partial r} + (\beta^2 - 2)\bar{e} - \beta^2 (1+\upsilon s)\bar{\theta} \tag{46a}$$

$$\bar{\sigma}_{r\varphi} = \frac{1}{r}\frac{\partial \bar{u}}{\partial \varphi} + \frac{\partial \bar{v}}{\partial r} - \frac{\bar{v}}{r} \tag{46b}$$

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Substituting from Eqs. (32), (33), (42) and (44) into Eqs.(46), we get

$$\bar{\sigma}_{rr} = 2(1+\upsilon s)\left\{\sum_{n=0}^{\infty} B_n(s)\left[\frac{n-1}{r^2} I_n(\beta s r) + \frac{\beta s}{r} I_{n+1}(\beta s r)\right]\cos(n\varphi) + \sum_{n=0}^{\infty}\sum_{i=1}^{2} A_{ni}\left[\left(\frac{n(n-1)}{r^2} + \frac{\beta^2 s^2}{2}\right) I_n(k_i r) - \frac{k_i}{r} I_{n+1}(k_i r)\right]\cos(n\varphi)\right\} + \bar{\sigma}_{rrpF} \tag{47}$$

$$\bar{\sigma}_{r\phi} = \frac{2}{r}(1+\upsilon s)\left\{\sum_{n=0}^{\infty} B_n(s)\left[\left(\frac{1-n}{r} - \frac{\beta^2 s^2 r}{2n}\right) I_n(\beta s r) + \frac{\beta s}{n} I_{n+1}(\beta s r)\right]\sin n\varphi + \sum_{n=0}^{\infty}\sum_{i=1}^{2} A_{ni}\left[\frac{n(1-n)}{r} I_n(k_i r) - n k_i I_{n+1}(k_i r)\right]\sin n\varphi\right\} + \bar{\sigma}_{r\varphi pF} \tag{48}$$

4. NUMERICAL RESULTS

In order to evaluate the unknown parameters A_{ni} , i =1,2 , $n = 1,2,.........$ and $B_n(s)$, $n = 1,2,.........$ we shall use the boundary conditions (19)

We expand the function $\bar{f}(\varphi, s)$ in Fourier cosine series in φ as

$$\bar{f}(\varphi, s) = \sum_{n=0}^{\infty} F_n(s)\cos n\varphi \tag{49}$$

where $F_0 = \frac{1}{\pi}\int_0^{\pi} \bar{f}(\varphi, s)\, d\varphi$, $F_n = \frac{1}{\pi}\int_0^{\pi} \bar{f}(\varphi, s)\cos n\varphi\, d\varphi$

Using Fourier cosine series, we get

$$F_0 = \frac{\varphi_0}{\pi} \; and \; F_n = \frac{2}{n\pi}\sin n\varphi_0 \qquad n = 1, 2, 3, ...$$

Also, we expand F_r , F_φ into Fourier series as follow:

$$F_r(r, \varphi, s) = \sum_{n=0}^{\infty} F_{rn}(r, s)\cos n\varphi$$

$$F_\varphi(r, \varphi, s) = \sum_{n=0}^{\infty} F_{\varphi n}(r, s)\sin n\varphi$$

where, F_{rn} , $F_{\varphi n}$ are the Fourier coefficients of F_r , F_φ respectively.

Form now on, we shall choose the components of the body force as follow,

$$F_r = Ar^2 \cos\varphi \tag{50a}$$

$$F_\varphi = 0 \tag{50b}$$

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$$\mathrm{F_z} = 0 \tag{50c}$$

where A is constant .

We have chosen these components to satisfy the condition that $(div\,\mathbf{F} \neq 0)$, to the authors knowledge, all pervious works in thermoelasticity which deal with external body forces always assumed that these forces were solenoidal $(div\,\mathbf{F} = 0)$ .This was done to simplify the governing equations.

We have,

$$div\mathbf{F} = \frac{1}{r}\left[\frac{\partial}{\partial r}(rF_r) + \frac{\partial F_\phi}{\partial \phi}\right] = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(\frac{rF_0}{2} + \sum_{n=1}^{\infty} rF_{rn}\cos n\phi\right) + \frac{\partial}{\partial \phi}\left(\sum_{n=1}^{\infty} F_{\phi n}\sin n\phi\right)\right] \tag{51}$$

For our choice,

$F_{r0} = 0, F_{r1} = Ar^2 \ and \ F_{r2} = F_{r3} = ... = 0$, then

$$\mathrm{div}\mathbf{F} = 3\mathrm{Ar}\varepsilon\cos\varphi \tag{52}$$

Substituting from (52) into (32),(33), (42), (45), (47) and (48) we get

$$\bar{\theta} = \sum_{n=0}^{\infty}\sum_{i=1}^{2} A_{ni}(k_i^2 - s^2)\mathrm{I}_n(k_i r)\cos(n\varphi) - \frac{3A\varepsilon}{s^3(1+\tau s)} r\cos\varphi \tag{53}$$

$$\bar{e} = (1+\upsilon s)\sum_{n=0}^{\infty}\sum_{i=1}^{2} A_{ni}k_i^2 I_n(k_i r)\cos(n\varphi) + \frac{3A}{s^3} r\cos\varphi \tag{54}$$

$$\bar{u} = \frac{(1+\upsilon s)}{r}\left[\sum_{n=1}^{\infty} B_n(s)\mathrm{I}_n(\beta s r)\cos n\varphi + \sum_{n=0}^{\infty}\sum_{i=1}^{2} A_{ni}\{nI_n(k_i r) + r\,k_i\, I_{n+1}(k_i r)\}\cos(n\varphi)\right] + \frac{Ar^3\cos\varphi}{s^3} + \frac{3\beta^2(1+\upsilon s)\varepsilon + (3\beta^2 - 1)(1+\tau s)}{\beta^2 s^5(1+\tau s)} Ar\cos\varphi \tag{55}$$

$$\bar{v} = -\frac{(1+\upsilon s)}{r}\left(\sum_{n=1}^{\infty} B_n(s)[\mathrm{I}_n(\beta s r) + \frac{\beta s r}{n} I_{n+1}(\beta s r)]\sin n\varphi + \sum_{n=0}^{\infty}\sum_{i=1}^{2} A_{ni} n I_n(k_i r)\sin n\varphi\right) - \frac{3\beta^2(1+\upsilon s)\varepsilon + (3\beta^2 - 1)(1+\tau s)}{\beta^2 s^5(1+\tau s)} A\sin\varphi \tag{56}$$

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$$\bar{\sigma}_{rr}=2(1+\upsilon s)\left\{\sum_{n=1}^{\infty}B_n(s)[\frac{n-1}{r^2}I_n(\beta sr)+\frac{\beta s}{r}I_{n+1}(\beta sr)]\cos n\varphi\right.$$
$$\left.+\sum_{n=0}^{\infty}\sum_{i=1}^{2}A_{ni}\{[\frac{n(n-1)}{r^2}+\frac{\beta^2 s^2}{2}]\mathrm{I}_n(k_i r)-\frac{k_i}{r}I_{n+1}(k_i r)\}\cos(n\varphi)\right\} \tag{57}$$
$$+\frac{3\beta^2(1+\upsilon s)\varepsilon+(3\beta^2-2)(1+\tau s)}{s^3(1+\tau s)}Ar\cos\varphi$$

$$\bar{\sigma}_{r\varphi}=\frac{2}{r}(1+\upsilon s)\left\{\sum_{n=1}^{\infty}B_n(s)\left([\frac{1-n}{r}-\frac{\beta^2 s^2 r}{2n}]\mathrm{I}_n(\beta sr)+\frac{\beta s}{n}I_{n+1}(\beta sr)\right)\sin n\varphi\right.$$
$$\left.+\sum_{n=0}^{\infty}\sum_{i=1}^{2}A_{ni}\left\{\frac{n(1-n)}{r}I_n(k_i r)-nk_i I_{n+1}(k_i r)\right\}\sin n\varphi\right\}-\frac{A}{s^3}r\sin\varphi \tag{58}$$

Using the boundary conditions (19) together with equations (57) and (53) for $n=0$ we get upon equating the coefficients of $\cos n\phi$ and $\sin n\phi$ the following.

$$A_{01}\{[\frac{\beta^2 s^2}{2}]\mathrm{I}_0(k_1 a)-\frac{k_1}{a}I_1(k_1 a)\}+A_{02}\{[\frac{\beta^2 s^2}{2}]\mathrm{I}_0(k_2 a)-\frac{k_2}{a}I_1(k_2 a)\}=0 \tag{59}$$

$$\mathrm{A}_{01}(\mathrm{k}_1^2-\mathrm{s}^2)\mathrm{I}_0(\mathrm{k}_1\mathrm{a})+\mathrm{A}_{02}(\mathrm{k}_2^2-\mathrm{s}^2)\mathrm{I}_0(\mathrm{k}_2\mathrm{a})=\mathrm{F}_0(\mathrm{s}) \tag{60}$$

For $n \succ 0$, equating the coefficients of $\cos n\varphi$ and $\sin n\varphi$ for both sides of equations (57), (58) and (53), we obtain

$$(1+\upsilon s)\left\{B_n(s)[\frac{n-1}{a^2}I_n(\beta sa)+\frac{\beta s}{a}I_{n+1}(\beta sa)]\right.$$
$$\left.+\sum_{i=1}^{2}A_{ni}\{[\frac{n(n-1)}{a^2}+\frac{\beta^2 s^2}{2}]\mathrm{I}_n(k_i a)-\frac{k_i}{a}I_{n+1}(k_i a)\}\right\} \tag{61}$$
$$=-\frac{3\beta^2(1+\upsilon s)\varepsilon+(3\beta^2-2)(1+\tau s)}{2s^3(1+\tau s)}Aa\delta_{n1}$$

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$$(1+\upsilon s)\left\{B_n(s)\left([\frac{1-n}{a}-\frac{\beta^2 s^2 a}{2n}]\mathrm{I}_n(\beta sa)+\frac{\beta s}{n}I_{n+1}(\beta sa)\right)\right.$$
$$\left.+\sum_{i=1}^{2}A_{ni}\left\{\frac{n(1-n)}{a}I_n(k_i a)-nk_i\,I_{n+1}(k_i a)\right\}\right\}=\frac{A}{2s^3}a^2\delta_{n1} \tag{62}$$

$$\sum_{i=1}^{2}A_{ni}(k_i^2-s^2)\mathrm{I}_n(k_i a)-\frac{3A\varepsilon}{s^3(1+\tau s)}a\delta_{n1}=F_n(s)\delta_{n1} \tag{63}$$

Solving the equations (59) – (63) numerically, we get the complete solution of the problem in transformed domain.

The constants of the problem are shown in table 1

Table 1: The material parameters

ρ = 8954 kg/m^3	α_t = 1.78 (10)$^{-5}$ K^{-1}	c_E = 381 J/(kg K)	η = 8886.73
μ =3.86 (10)^{10} kg/(m s^2)	λ = 7.76 (10)^{10} kg/(m s^2)	ε = 0.0168	T_0 = 293 K
a = 1 m	τ = 0.02 s	A=1	$\upsilon = 0.02\ s$

The copper material was chosen for purposes of numerical evaluations. The constants of the problem are shown in Table 1.

The surface of the cylinder is kept at a constant temperature equal to unity over the sector $-\varphi_0 \le \varphi \le \varphi_0$ and zero everywhere else. The constant φ_0 was taken as $\dfrac{\pi}{12}$ during computation.

The Fourier coefficients are thus given by $F_0=\dfrac{\varphi_0}{\pi}$ and $F_n=\dfrac{2}{n\pi}\sin n\varphi_0, \quad n=1,2,3,......$

The problem was solved above in the transformed domain in the form of a series of complex numbers. To obtain the solution in the physical domain, we have tried two different approaches.

(1) The series of complex terms was summed to give a complex functions. This function was inverted using a numerical approach whose details can be found in [24].

(2) Each term of the series was inverted using the above mentioned method and then the series was summed.

It was found that the second method is much better in terms of the accuracy achieved and run time of the program.

We used the Fortran programming language. The accuracy maintained was five digits. The maximum number of terms used varies according to the function used. This numbers was 50 for the temperature, 20 for the displacement and 40 for the stress.

All the functions were carried out for two values of time $t=\{0.1, 0.2\}$.

The graphs of the temperature, displacement and stress are shown in figures 1-3, respectively plotted on the diagonal $\varphi=\{0,\pi\}$, while the temperature, displacement and stress are shown in figures 4-6, respectively plotted on the diagonal $\varphi=\{\pi/9, 10\pi/9\}$.

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5. CONCULUSION

We conclude from these figures the following:

(1) The effect of the body force on the temperature is very small.

(2) The effect is more pronounced in the case of the displacement and the stress.

(3) The waves for all the functions travel with finite speeds. The solution is identically zero far away from the source of disturbance.

(4) In this work, we have shown how to deal with non solenoidal body forces. To the authors' knowledge, this is the first work to deal with a body force not satisfying the condition that $div\overline{\mathbf{F}} = 0$.

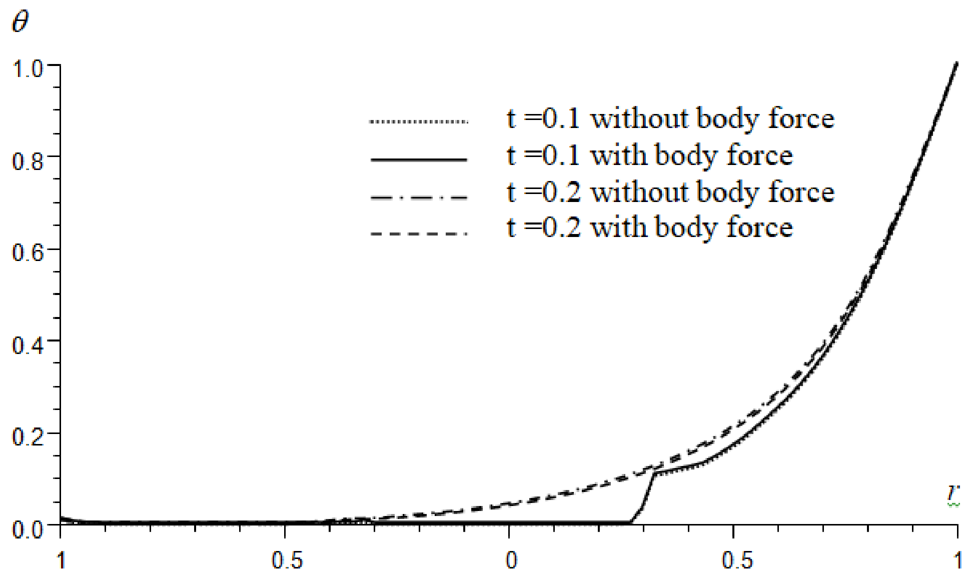

Fig. 1. Temperature distribution with the diagonal $\varphi = 0$.

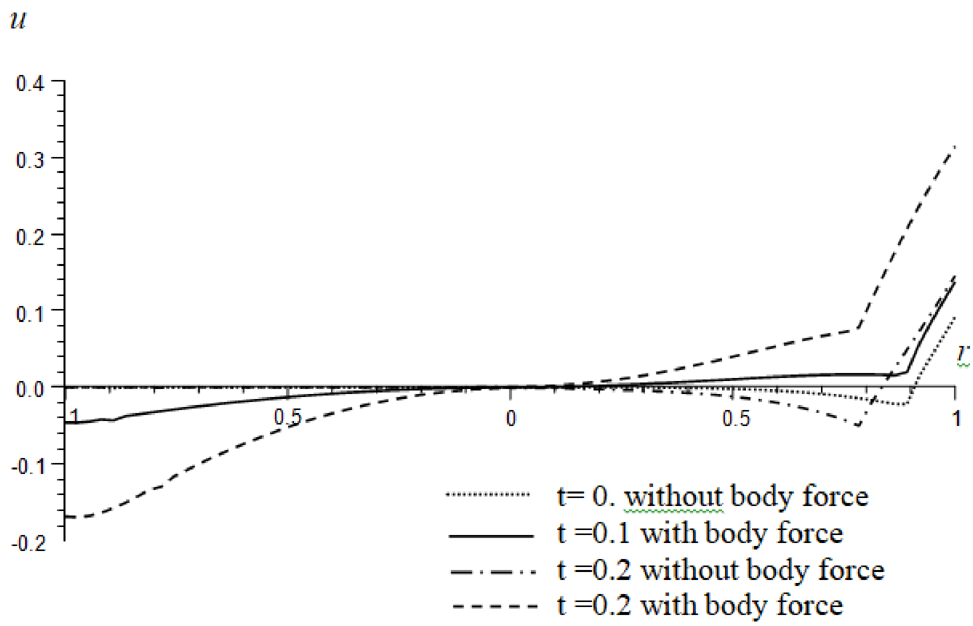

Fig. 2. Radial displacement distribution with the diagonal $\varphi = 0$.

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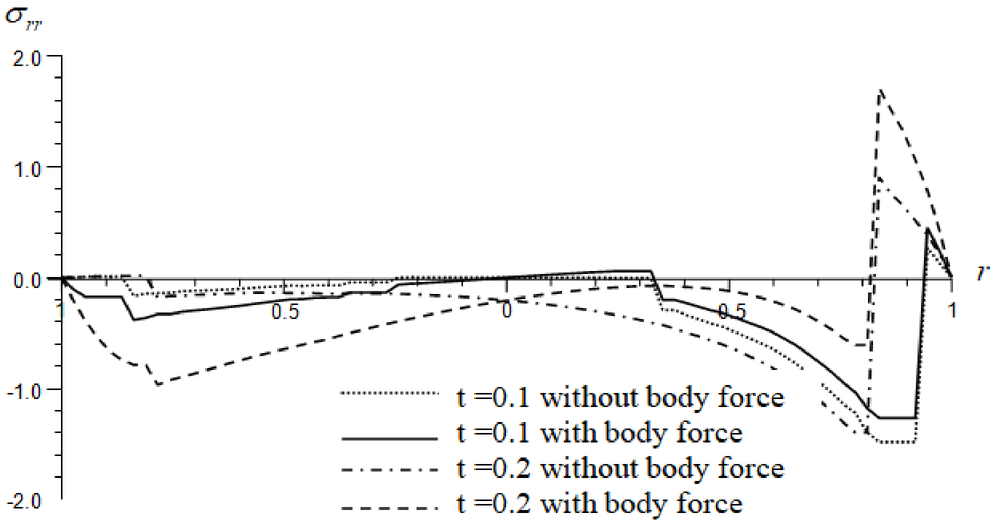

Fig. 3. Radial stress distribution with the diagonal $\varphi = 0$ **.**

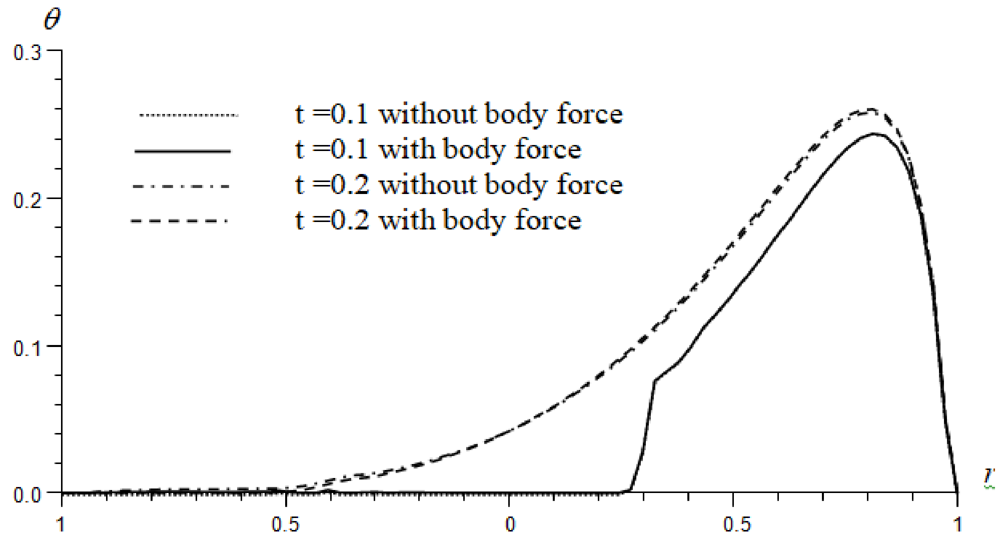

Fig. 4. Temperature distribution with the diagonal $\varphi = \pi/9$ **.**

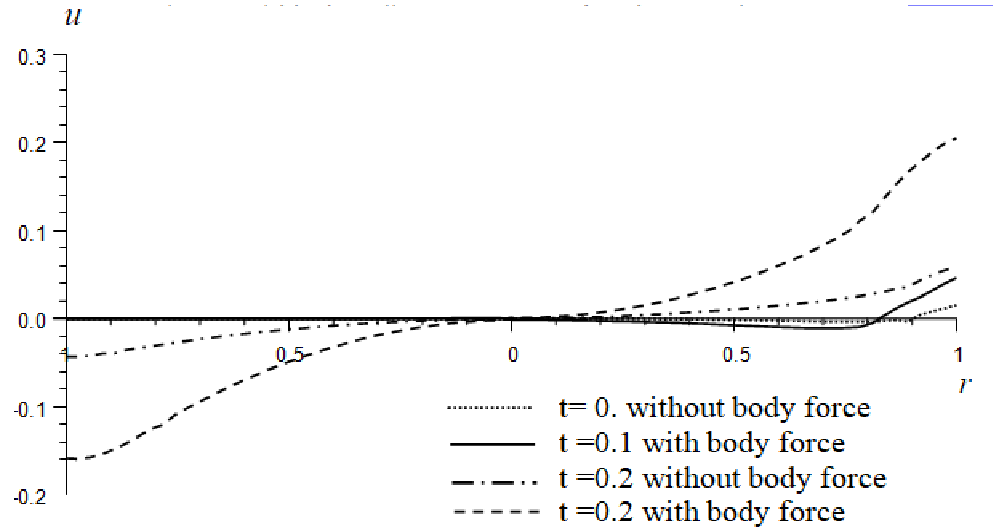

Fig. 5. Radial displacement distribution with the diagonal $\varphi = \pi/9$ **.**

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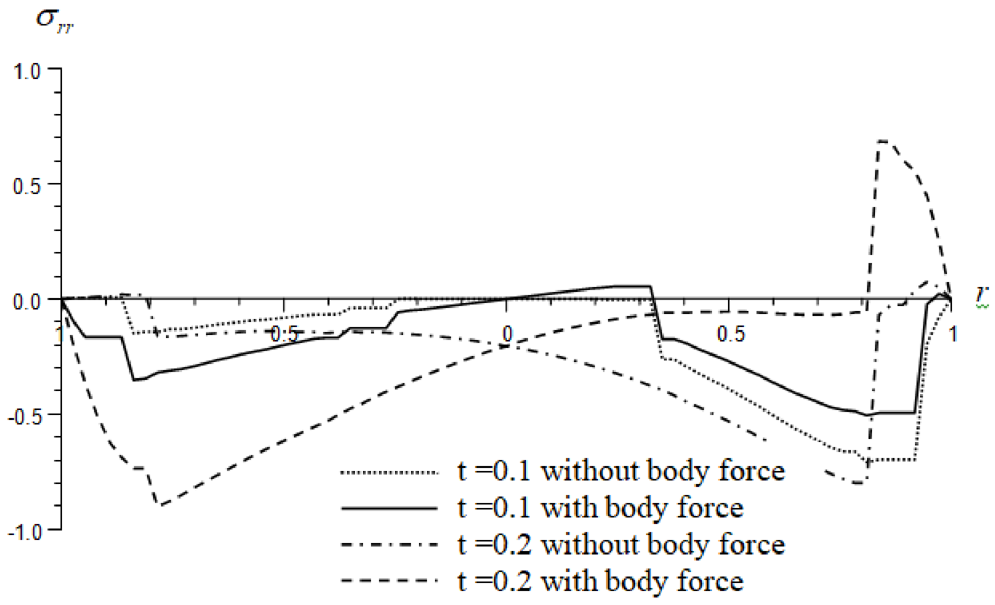

Fig. 6. Radial stress distribution with the diagonal $\varphi = \pi/9$.

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